Introduction to Graphs

CS 351: Analysis of Algorithms

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This lecture covers the introduction to graphs, including their definition, types, and applications. We will explore the different types of graphs, their properties, and their applications.

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0.1 Executive Summary

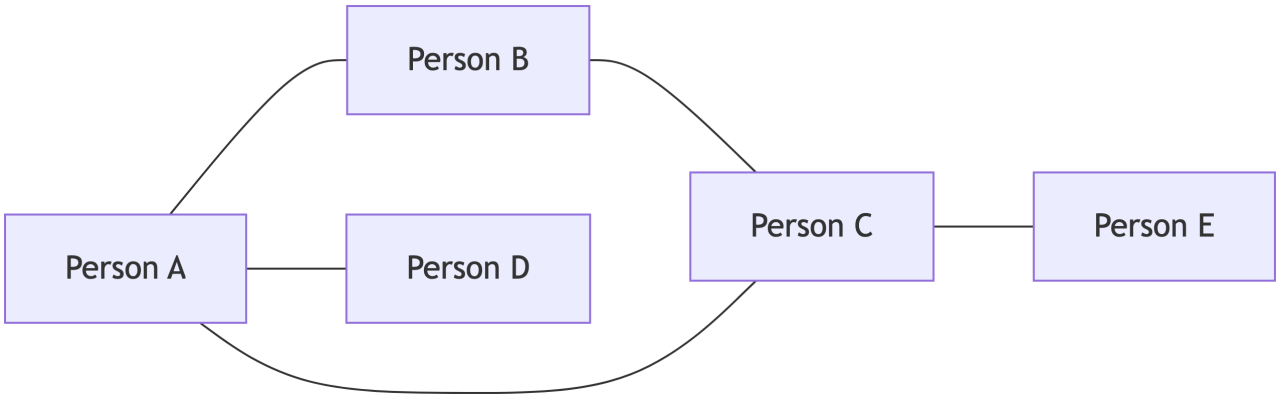
What We'll Discuss This Week

- Introduction to graphs
- Definition of graphs
- Types of graphs
- Applications of graphs

1 Introduction to Graphs

1.1 What Are Graphs?

When you hear "graph," you might think of x-y plots, but in computer science, graphs represent relationships between objects.



Key Insight: Graphs model pairwise relationships between objects

1.2 Two Types of "Graphs"

Mathematical Plot

- x-axis and y-axis
- Functions: f(n) = n, f(n) = log n
- Continuous relationships

Graph Data Structure

- Vertices (nodes) = objects

- Edges = relationships
- Discrete connections

2 Graph Fundamentals

2.1 Essential Vocabulary

Vertices (or Nodes):

- The objects being represented
- Examples: people, intersections, web pages

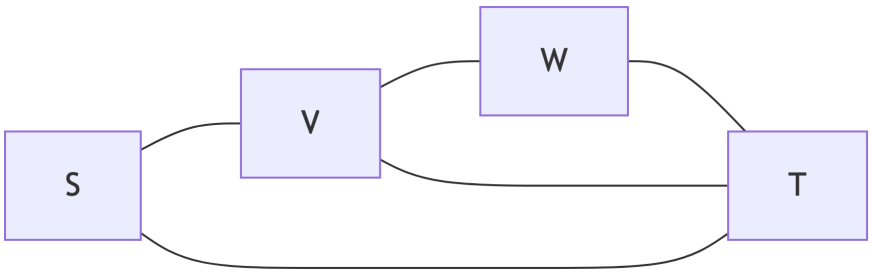
Edges:

- The pairwise relationships
- Examples: friendships, roads, hyperlinks

Notation: G = (V, E) where V = vertices, E = edges

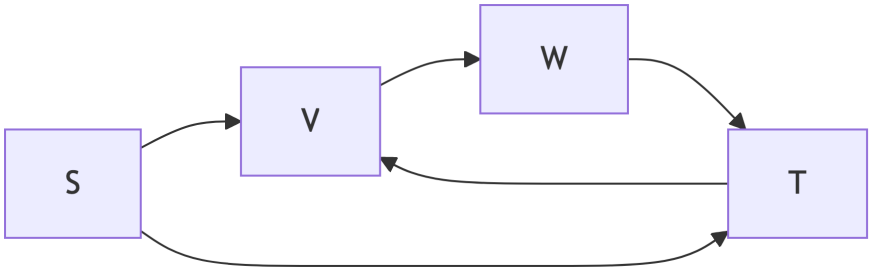
2.2 Directed vs Undirected Graphs

Undirected Graph



- Edges are unordered pairs {v, w}
- No direction matters
- (v, w) = (w, v)

Directed Graph



- Edges are ordered pairs (v, w)
- Direction matters: v → w
- v = tail, w = head

2.3 Graph Notation Deep Dive

For any graph G = (V, E):

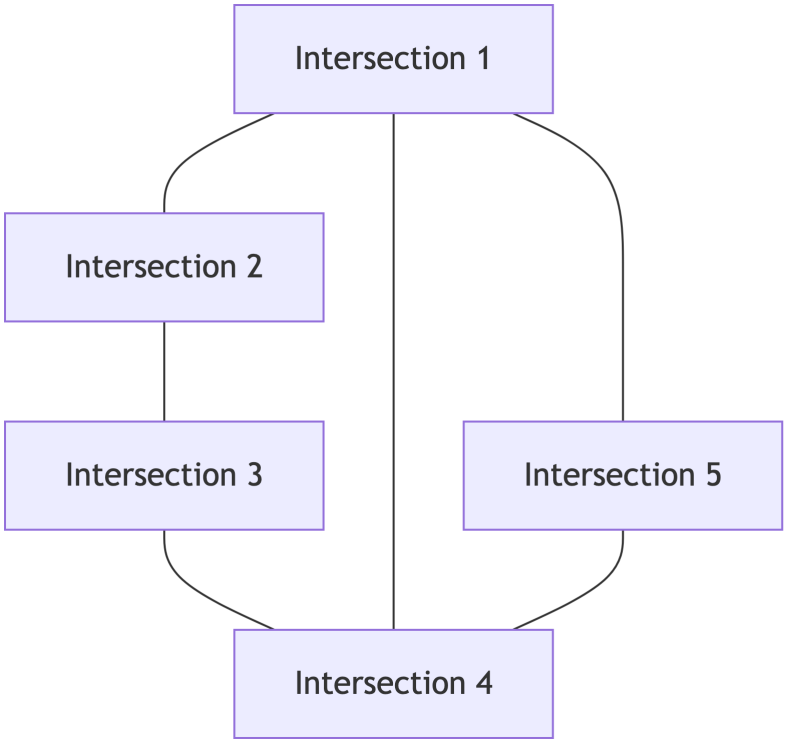
- |V| = n (number of vertices)
- |E| = m (number of edges)
- Vertices often labeled: v , v , …, v
- Edges connect vertex pairs

Example: Social network graph

- V = {Alice, Bob, Carol, Dave}
- E = {(Alice,Bob), (Bob,Carol), (Alice,Carol)}

3 Real-World Applications

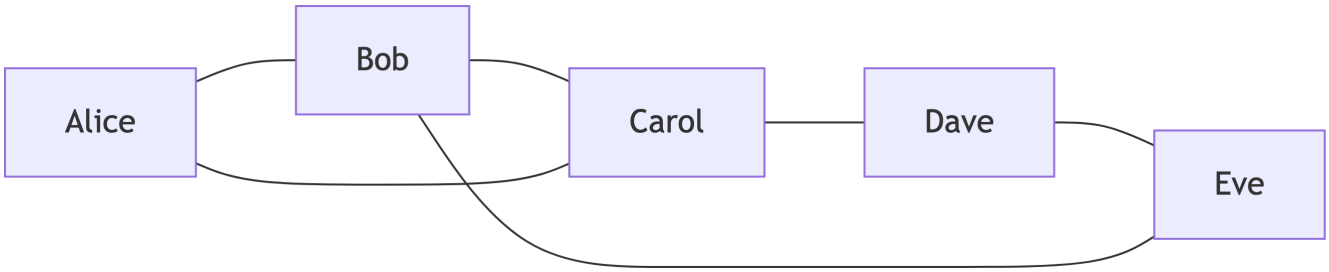
3.1 Road Networks



Implementation:

- Vertices = intersections
- Edges = road segments
- Used by GPS navigation systems

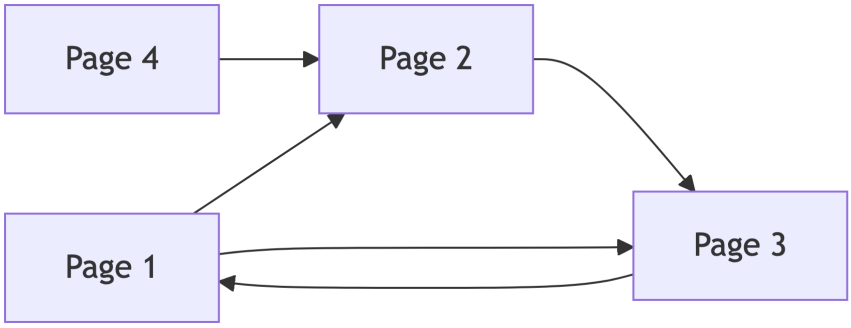
3.2 Social Networks



Applications:

- Friend recommendations
- Information spread analysis
- Community detection

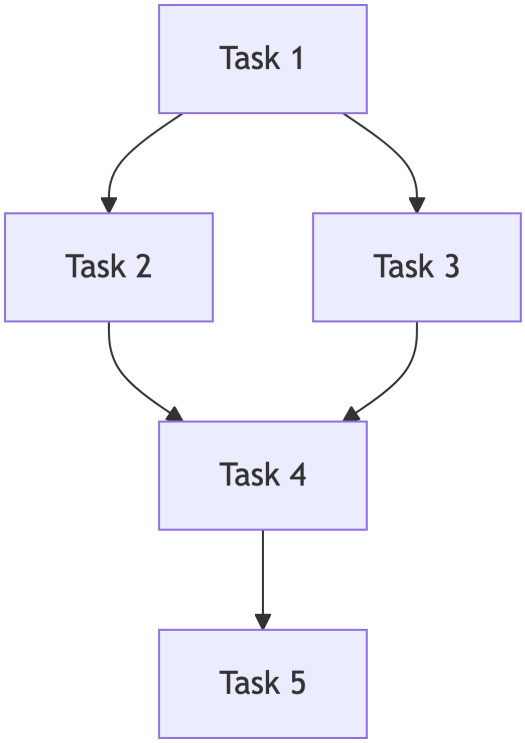
3.3 Web Structure



Key Features:

- Vertices = web pages
- Edges = hyperlinks
- Foundation of PageRank algorithm

3.4 Dependency Networks



Use Cases:

- Project scheduling
- Software compilation
- Course prerequisites

4 Graph Size Analysis

4.1 Measuring Graph Size

Two key metrics:

- n = number of vertices
- m = number of edges

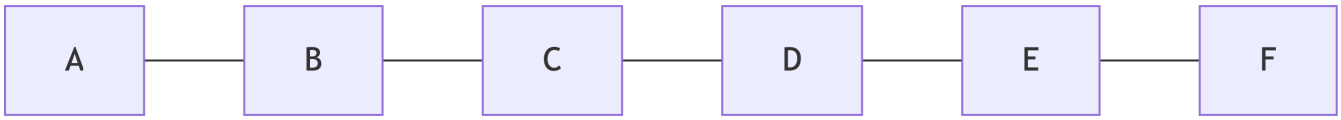
Fundamental bounds:

- Minimum edges: m 0

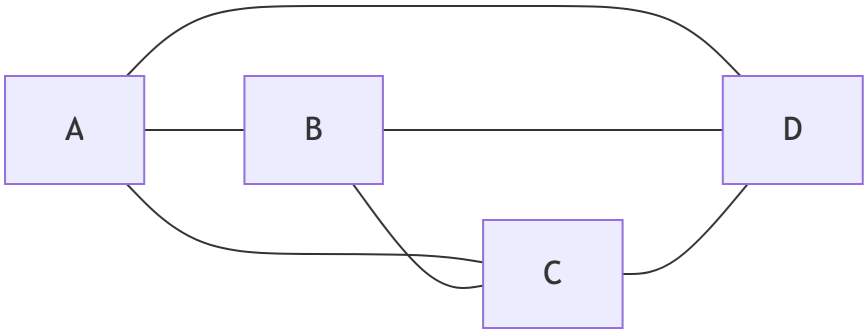
- Maximum edges (undirected): m n(n-1)/2
- Maximum edges (directed): m n(n-1)

4.2 Sparse vs Dense Graphs

Sparse Graph: m O(n)



Dense Graph: m $O(n^2)$



4.3 When Is a Graph Dense?

Mathematical Definition:

- Sparse: m = O(n) or m = O(n log n)
- Dense: m = $\Theta(n^2)$

Practical Examples:

- Social networks: typically sparse
- Complete graphs: always dense
- Road networks: usually sparse

5 Graph Representation

5.1 Two Main Approaches

Adjacency Lists:

- Space efficient for sparse graphs
- Fast edge iteration
- Preferred for most algorithms

Adjacency Matrix:

- Space efficient for dense graphs

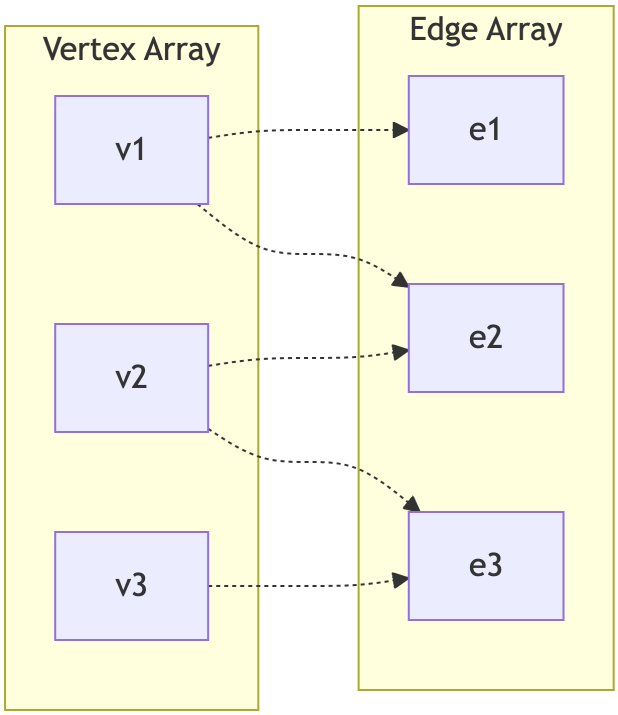
- Fast edge lookup
- Simple implementation

5.2 Adjacency List Representation

Core Components:

1. **Vertex Array**: stores vertex information
2. **Edge Array**: stores edge information

3. **Edge → Vertex pointers**: each edge points to endpoints
4. **Vertex → Edge pointers**: each vertex points to incident edges



5.3 Adjacency List: Space Analysis

Space Complexity: $\Theta(m + n)$

Breakdown:

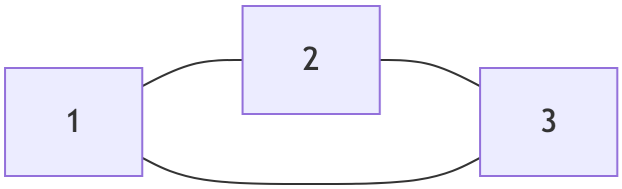
- Vertex array: $\Theta(n)$
- Edge array: $\Theta(m)$

- Edge→vertex pointers: $\Theta(m)$
- Vertex→edge pointers: $\Theta(m)$

Total: $\Theta(n + m + m + m) = \Theta(m + n)$

5.4 Adjacency List Example

Graph:



Adjacency List Representation:

- Vertex 1: [2, 3]
- Vertex 2: [1, 3]
- Vertex 3: [1, 2]

Each vertex stores list of neighbors

5.5 Adjacency Matrix Representation

Structure: n × n matrix A where:

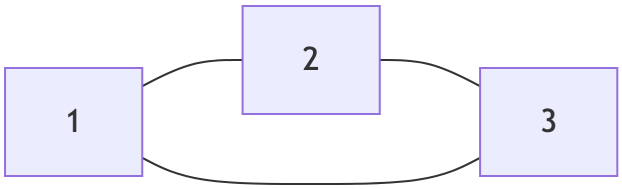
- A[i,j] = 1 if edge (i,j) exists
- A[i,j] = 0 if no edge

Undirected graphs: A[i,j] = A[j,i] (symmetric)

Directed graphs: A[i,j] A[j,i] generally

5.6 Adjacency Matrix Example

Graph:



Matrix Representation:

```
    1  2  3
1 [ 0  1  1 ]
2 [ 1  0  1 ]
3 [ 1  1  0 ]
```

5.7 Adjacency Matrix: Space Analysis

Space Complexity: $\Theta(n^2)$

Characteristics:

- Always uses n^2 bits regardless of edge count
- Efficient for dense graphs (m n^2)
- Wasteful for sparse graphs (m n)
- Fast edge existence queries: O(1)

6 Comparison and Trade-offs

6.1 Adjacency Lists vs Adjacency Matrix

Operation	Adjacency List	Adjacency Matrix
Space	$\Theta(m + n)$	$\Theta(n^2)$
Edge lookup	O(degree(v))	O(1)
Add edge	O(1)	O(1)
Remove edge	O(degree(v))	O(1)
Find neighbors	O(degree(v))	O(n)

6.2 When to Use Each Representation

Use Adjacency Lists when:

- Graph is sparse (m n)
- Need to iterate over neighbors frequently
- Memory is limited
- Most graph algorithms prefer this

Use Adjacency Matrix when:

- Graph is dense (m n^2)
- Need fast edge existence queries
- Working with mathematical graph operations
- Graph size is small and fixed

6.3 Implementation Considerations

Adjacency Lists:

- Dynamic arrays or linked lists
- Easy to add/remove vertices
- Cache-friendly for sparse graphs
- Standard choice for most algorithms

Adjacency Matrix:

- Fixed-size 2D array
- Simple bit operations
- Good for mathematical computations
- Parallel processing friendly

7 Quiz Time!

7.1 Quiz Question 1

For an undirected graph with n vertices, what is the maximum possible number of edges?

A) n

B) n - 1

C) n(n-1)/2

D) n^2

7.2 Quiz Question 2

Which representation is better for a sparse graph with 1000 vertices and 1500 edges?

A) Adjacency Matrix

B) Adjacency List

C) Both are equivalent

D) Depends on the operations

7.3 Quiz Question 3

In an adjacency list representation, how many operations are needed to check if edge (u,v) exists?

A) O(1)

B) O(log n)

C) O(degree(u))

D) O(n)

8 Practical Applications

8.1 Algorithm Performance Impact

Graph Search Algorithms:

- Breadth-First Search: O(m + n)
- Depth-First Search: O(m + n)

- Both prefer adjacency lists

Shortest Path Algorithms:

- Dijkstra's: O((m + n) log n) with heaps
- Works with both representations
- Adjacency lists usually faster

8.2 Memory Usage in Practice

Real-world example: Facebook social graph

- ~3 billion users (vertices)
- ~200 billion friendships (edges)
- Average degree: ~130
- Adjacency matrix: 9×10^1 bits 1 exabyte
- Adjacency list: ~400 billion entries 1.6 TB

Clear winner: Adjacency lists!

8.3 Graph Processing Libraries

Common implementations:

- NetworkX (Python): adjacency lists
- igraph (R/Python): adjacency lists

- GraphX (Spark): distributed adjacency lists
- SNAP (C++): adjacency lists

Industry standard: Adjacency lists with optimizations

9 Advanced Topics Preview

9.1 What's Next?

Graph Search Algorithms (Chapter 8):

- Breadth-First Search (BFS)
- Depth-First Search (DFS)
- Connected Components
- Topological Sort

Shortest Path Algorithms (Chapter 9):

- Dijkstra's Algorithm
- Single-source shortest paths
- Applications to routing

9.2 Graph Algorithm Complexity

All upcoming algorithms are "for-free primitives":

- Running time: O(m + n)
- Just slightly more than reading input
- Can apply liberally to understand graph structure

Key insight: Linear-time graph algorithms are incredibly powerful!

9.3 Modern Graph Challenges

Big Data Graphs:

- Distributed storage and computation
- Streaming graph algorithms
- Approximate algorithms for massive graphs

Machine Learning on Graphs:

- Graph Neural Networks
- Node embeddings
- Graph classification

10 Summary

10.1 Key Takeaways

Graph Fundamentals:

- Graphs model pairwise relationships
- Directed vs undirected matters
- Vertices = objects, Edges = relationships

Representation Trade-offs:

- Adjacency lists: $\Theta(m + n)$ space, prefer for sparse graphs
- Adjacency matrix: $\Theta(n^2)$ space, prefer for dense graphs

- Most algorithms use adjacency lists

10.2 Design Principles

When designing with graphs:

1. **Identify** vertices and edges clearly
2. **Choose** representation based on density
3. **Consider** the operations you'll need most
4. **Remember** adjacency lists are usually best
5. **Think** about scalability early

10.3 Next Steps

Build on this foundation:

- Practice implementing both representations
- Study graph search algorithms
- Explore real-world graph datasets
- Consider distributed graph processing
- Learn graph visualization techniques