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Slide Notes or Announcements

None

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Introduction to Algorithms

Alternative Formats

- **Slides**
- **Outline**
- **PDF**

Part 1: Course Introduction & Why Algorithms Matter

Welcome to Algorithms! 🚀

What We’ll Learn

- **Design** algorithms that solve problems
- **Analyze** their efficiency
- **Compare** different approaches
- **Apply** algorithmic thinking

Course Philosophy

> “Perhaps the most important principle for the good algorithm designer is to **refuse to be content**.”
>
> — Aho, Hopcroft, and Ullman

Our Mantra: Can we do better? 🤔

Why Study Algorithms?

1. Foundation for Computer Science 🏗️

- **Routing protocols** → Shortest path algorithms
- **Cryptography** → Number-theoretic algorithms
- **Graphics** → Geometric algorithms

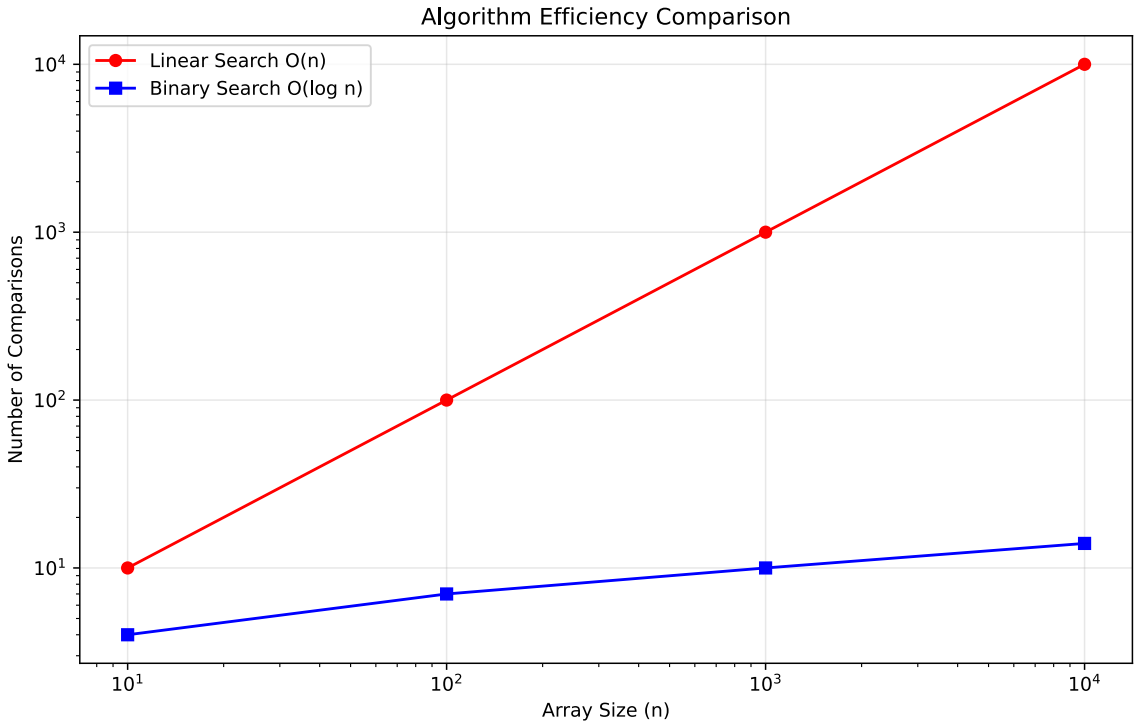
- **Databases** → Search tree data structures
- **Biology** → Dynamic programming for genome similarity

2. Driver of Innovation 💡
"Performance gains due to improvements in algorithms have vastly exceeded even the dramatic performance gains due to increased processor speed."
— *Report to the White House, 2010*

Interactive Demo: The Power of Algorithms

```
import time
import matplotlib.pyplot as plt
import numpy as np

# Simulating algorithm performance
def naive_search(arr, target):
    """O(n) linear search"""
    comparisons = 0
    for i in range(len(arr)):
        comparisons += 1
        if arr[i] == target:
            return comparisons
    return comparisons

def binary_search(arr, target):
    """O(log n) binary search"""
    comparisons = 0
    left, right = 0, len(arr) - 1
    while left <= right:
        comparisons += 1
        mid = (left + right) // 2
        if arr[mid] == target:
            return comparisons
        elif arr[mid] < target:
            left = mid + 1
        else:
            right = mid - 1
    return comparisons

# Visualization
sizes = [10, 100, 1000, 10000]
linear_comps = []
binary_comps = []

for n in sizes:
    arr = list(range(n))
    linear_comps.append(naive_search(arr, n-1))
    binary_comps.append(binary_search(arr, n-1))
```

```
plt.figure(figsize=(10, 6))
plt.plot(sizes, linear_comps, 'r-', label='Linear Search O(n)', marker='o')
plt.plot(sizes, binary_comps, 'b-', label='Binary Search O(log n)',
marker='s')
plt.xlabel('Array Size (n)')
plt.ylabel('Number of Comparisons')
plt.title('Algorithm Efficiency Comparison')
plt.legend()
plt.xscale('log')
plt.yscale('log')
plt.grid(True, alpha=0.3)
plt.show()
```



🎯 Quick Quiz 1

Question: You have 1 million sorted numbers. How many comparisons does binary search need in the worst case?

Answer: About 20 comparisons!

```
- $\log_2(1,000,000) \approx 20$
- Linear search would need 1,000,000 comparisons!
```

Integer Multiplication: A First Example

The Problem

- **Input:** Two n-digit numbers, x and y
- **Output:** Their product x · y

Grade-School Algorithm

```
   5678
 × 1234
 ------
  22712  (5678 × 4)
 17034   (5678 × 3, shifted)
 11356   (5678 × 2, shifted)
 5678    (5678 × 1, shifted)
 ------
7006652
```

Time Complexity: ~n^2 operations

Can we do better? 🤔

Karatsuba's Breakthrough (1960)

The Clever Trick 🎩

Split numbers in half: $x = 10^{n/2} \cdot a + b$

Naive approach: 4 multiplications - $a \cdot c$ - $a \cdot d$
- $b \cdot c$ - $b \cdot d$

Karatsuba: Only 3 multiplications! 1. $P_1 = a \cdot c$ 2. $P_2 = b \cdot d$ 3. $P_3 = (a + b) \cdot (c + d)$

Then: $ad + bc = P_3 - P_1 - P_2$

Live Coding: Karatsuba Implementation

```python
def karatsuba(x, y):
    """Karatsuba multiplication algorithm"""
    # Base case
    if x < 10 or y < 10:
        return x * y

    # Calculate size of numbers
    n = max(len(str(x)), len(str(y)))
    half = n // 2

    # Split the numbers
    a = x // (10 ** half)
    b = x % (10 ** half)
    c = y // (10 ** half)
```

```
    d = y % (10 ** half)

    # Three recursive calls
    ac = karatsuba(a, c)
    bd = karatsuba(b, d)
    ad_plus_bc = karatsuba(a + b, c + d) - ac - bd

    # Combine results
    return ac * (10 ** (2 * half)) + ad_plus_bc * (10 ** half) + bd

# Test it!
x = 1234
y = 5678
result = karatsuba(x, y)
print(f"{x} × {y} = {result}")
print(f"Verification: {x * y == result}")
```

```
1234 × 5678 = 7006652
Verification: True
```

Part 2: Algorithm Fundamentals with MergeSort

The Sorting Problem

Definition

- **Input:** Array of n numbers in arbitrary order
- **Output:** Same numbers, sorted from smallest to largest

Example

```
Input:  [5, 4, 1, 8, 7, 2, 6, 3]
Output: [1, 2, 3, 4, 5, 6, 7, 8]
```

Simple Approaches

- **SelectionSort:** Find minimum, repeat → $O(n^2)$
- **InsertionSort:** Build sorted portion → $O(n^2)$
- **BubbleSort:** Swap adjacent pairs → $O(n^2)$

Can we do better? 🚀

Divide and Conquer Paradigm

The Strategy

1. **Divide** the problem into smaller subproblems
2. **Conquer** the subproblems recursively
3. **Combine** the solutions

MergeSort Overview

```
{mermaid}
graph TD
    A[5,4,1,8,7,2,6,3] --> B[5,4,1,8]
    A --> C[7,2,6,3]
    B --> D[5,4]
    B --> E[1,8]
    C --> F[7,2]
    C --> G[6,3]
    D --> H[1,4,5,8]
    E --> H
    F --> I[2,3,6,7]
    G --> I
    H --> J[1,2,3,4,5,6,7,8]
    I --> J
```

Interactive MergeSort Visualization

```python
def merge(left, right):
    """Merge two sorted arrays"""
    result = []
    i = j = 0

    # Merge process with step tracking
    steps = []
    while i < len(left) and j < len(right):
        if left[i] <= right[j]:
            result.append(left[i])
            steps.append(f"Compare {left[i]} ≤ {right[j]}: Take {left[i]} from
left")
            i += 1
        else:
            result.append(right[j])
            steps.append(f"Compare {left[i]} > {right[j]}: Take {right[j]}
from right")
            j += 1

    # Add remaining elements
    result.extend(left[i:])
    result.extend(right[j:])

    return result, steps

def mergesort_visual(arr, depth=0):
    """MergeSort with visualization"""
    indent = "  " * depth
    print(f"{indent}Sorting: {arr}")
```

```python
    if len(arr) <= 1:
        return arr

    mid = len(arr) // 2
    left = mergesort_visual(arr[:mid], depth + 1)
    right = mergesort_visual(arr[mid:], depth + 1)

    result, steps = merge(left, right)
    print(f"{indent}Merging {left} + {right} = {result}")

    return result

# Demo
test_array = [5, 4, 1, 8, 7, 2, 6, 3]
print("MergeSort Trace:")
sorted_array = mergesort_visual(test_array)
print(f"\nFinal result: {sorted_array}")
```

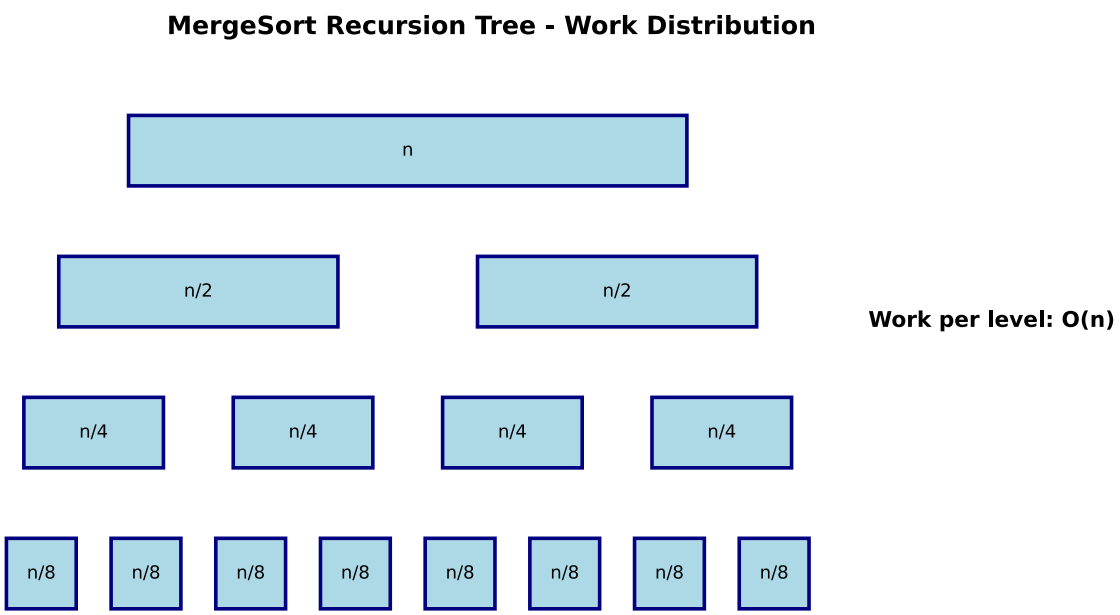
```
MergeSort Trace:
Sorting: [5, 4, 1, 8, 7, 2, 6, 3]
  Sorting: [5, 4, 1, 8]
    Sorting: [5, 4]
      Sorting: [5]
      Sorting: [4]
    Merging [5] + [4] = [4, 5]
    Sorting: [1, 8]
      Sorting: [1]
      Sorting: [8]
    Merging [1] + [8] = [1, 8]
  Merging [4, 5] + [1, 8] = [1, 4, 5, 8]
  Sorting: [7, 2, 6, 3]
    Sorting: [7, 2]
      Sorting: [7]
      Sorting: [2]
    Merging [7] + [2] = [2, 7]
    Sorting: [6, 3]
      Sorting: [6]
      Sorting: [3]
    Merging [6] + [3] = [3, 6]
  Merging [2, 7] + [3, 6] = [2, 3, 6, 7]
Merging [1, 4, 5, 8] + [2, 3, 6, 7] = [1, 2, 3, 4, 5, 6, 7, 8]

Final result: [1, 2, 3, 4, 5, 6, 7, 8]
```

Analyzing MergeSort: Recursion Tree

Key Insights 🔍
At each level j: - 2^j subproblems - Each of size $n/2^j$ - Work per level: $O(n)$

Total analysis: - Tree depth: $\log_2 n$ - Levels: $\log_2 n + 1$ - Total work: $O(n \log n)$

```
# Visualizing recursion tree work
import matplotlib.pyplot as plt

levels = 4
fig, ax = plt.subplots(figsize=(10, 6))

for level in range(levels):
    num_nodes = 2**level
    node_size = 100 / num_nodes
    work = f"n/{2**level}" if level > 0 else "n"

    for node in range(num_nodes):
        x = (node + 0.5) * (100 / num_nodes)
        y = (levels - level - 1) * 20

        # Draw node
        rect = plt.Rectangle((x - node_size/3, y), node_size/1.5, 10,
                             fill=True, facecolor='lightblue',
                             edgecolor='navy', linewidth=2)
        ax.add_patch(rect)
        ax.text(x, y + 5, work, ha='center', va='center', fontsize=10)

ax.text(105, levels * 10, f"Work per level: O(n)", fontsize=12, weight='bold')
ax.set_xlim(0, 120)
ax.set_ylim(-5, levels * 20)
ax.axis('off')
ax.set_title('MergeSort Recursion Tree - Work Distribution', fontsize=14,
weight='bold')
plt.show()
```



🎯 Quick Quiz 2

Question: If MergeSort takes 1 second to sort 1 million items, approximately how long for 2 million items?

Answer: About 2.02 seconds! - Time complexity: $O(n \log n)$ - $2n \log(2n) = 2n(\log n + 1) \approx 2n \log n + 2n$ - Roughly doubles (plus a tiny bit more)

Hands-On Exercise: Trace MergeSort

Your Turn! Trace MergeSort on: `[3, 7, 1, 4]`

1. Draw the recursion tree
2. Show the merge steps
3. Count total comparisons

Solution:

```
      [3,7,1,4]
      /       \
   [3,7]      [1,4]
   /   \      /   \
 [3]   [7]  [1]   [4]
   \   /      \   /
   [3,7]      [1,4]
      \       /
      [1,3,4,7]
```

Total comparisons: ~5

Part 3: Big-O Notation & Asymptotic Analysis

Why Asymptotic Notation?

The Challenge

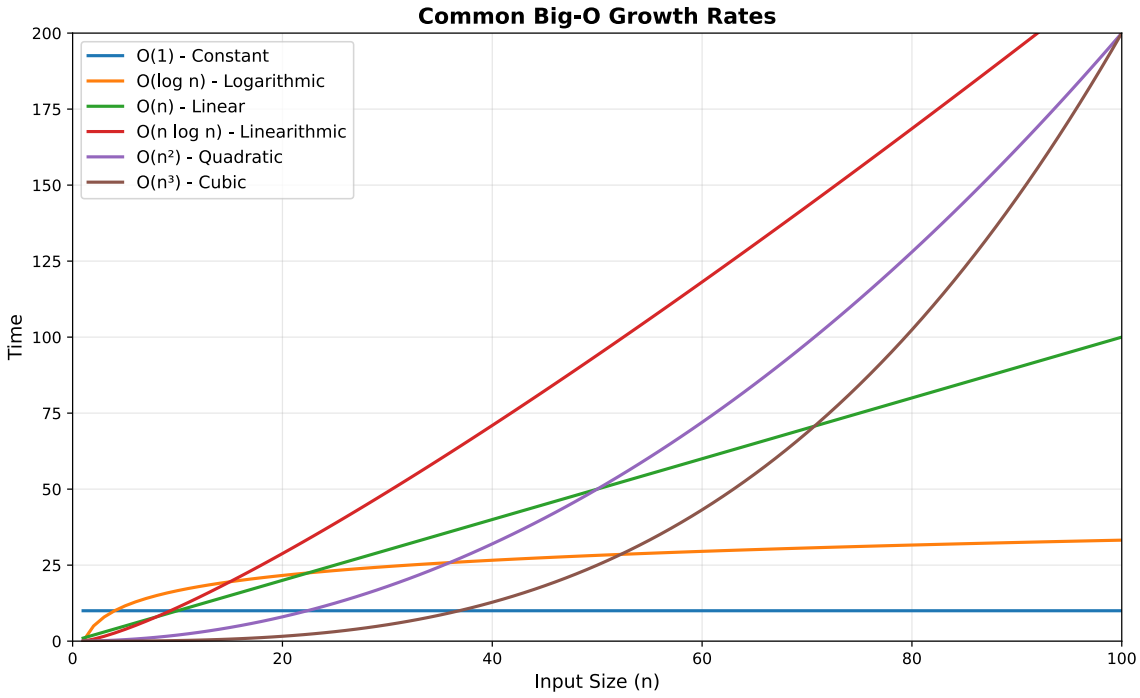
How do we compare algorithms across: - Different programming languages? - Different hardware? - Different compilers? - Different input sizes?

The Solution: Big-O Notation

Focus on growth rate, not exact counts!

Suppress: - Constant factors (system-dependent) - Lower-order terms (irrelevant for large n)

Big-O Intuition



Formal Definition of Big-O

Mathematical Definition

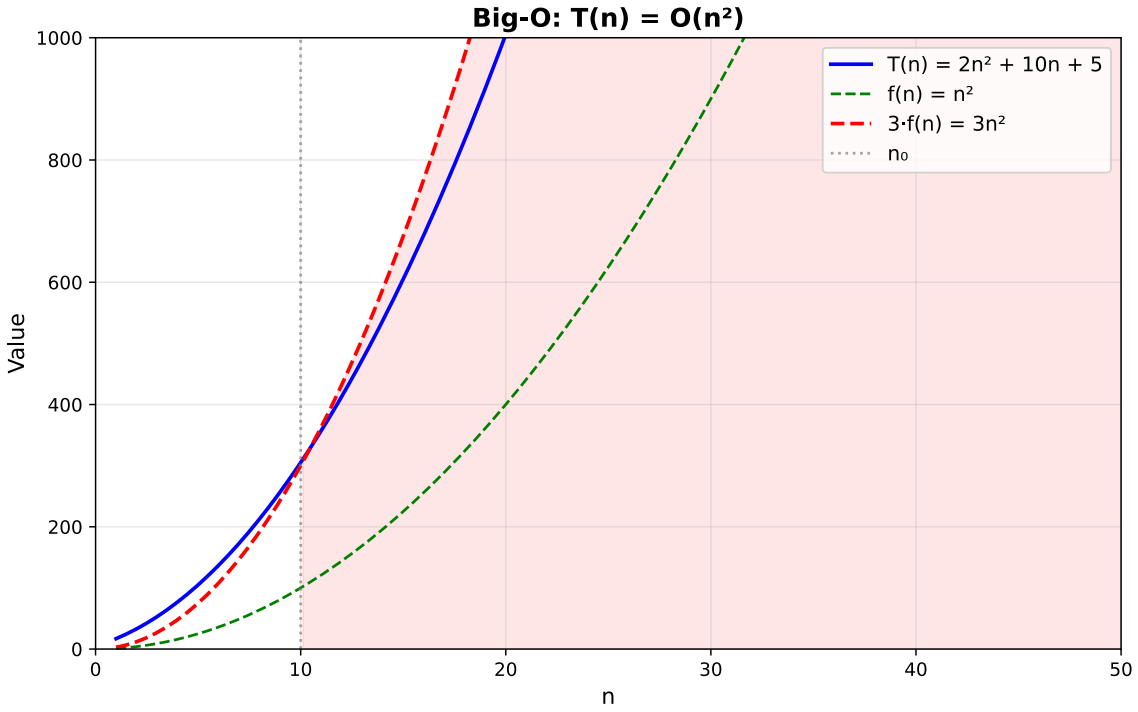
$T(n) = O(f(n))$ if and only if there exist positive constants c and n_0 such that:

$$T(n) \leq c \cdot f(n) \quad \text{for all} \quad n \geq n_0$$

In Plain English

"$T(n)$ is eventually bounded above by a constant multiple of $f(n)$"

Visual Interpretation



Interactive Big-O Practice

```
def analyze_code_complexity(code_type):
    """Analyze different code patterns"""

    examples = {
        'constant': {
            'code': 'x = arr[0] + arr[n-1]',
            'complexity': 'O(1)',
            'explanation': 'Fixed number of operations'
        },
        'linear': {
            'code': 'for i in range(n):\n    process(arr[i])',
            'complexity': 'O(n)',
            'explanation': 'Single loop through n elements'
        },
        'quadratic': {
            'code': 'for i in range(n):\n    for j in range(n):\n
process(arr[i], arr[j])',
            'complexity': 'O(n²)',
            'explanation': 'Nested loops, n × n iterations'
        },
        'logarithmic': {
            'code': 'while n > 1:\n    n = n // 2\n    process()',
```

```
            'complexity': 'O(log n)',
            'explanation': 'Halving n each iteration'
        }
    }

    example = examples.get(code_type, examples['linear'])
    print(f"Code Pattern:\n{example['code']}\n")
    print(f"Complexity: {example['complexity']}")
    print(f"Reason: {example['explanation']}")
    return example

# Try different patterns
print("=" * 50)
print("ANALYZING CODE COMPLEXITY")
print("=" * 50)
for pattern in ['constant', 'logarithmic', 'linear', 'quadratic']:
    analyze_code_complexity(pattern)
    print("-" * 50)
```

```
==================================================
ANALYZING CODE COMPLEXITY
==================================================
Code Pattern:
x = arr[0] + arr[n-1]

Complexity: O(1)
Reason: Fixed number of operations
--------------------------------------------------
Code Pattern:
while n > 1:
    n = n // 2
    process()

Complexity: O(log n)
Reason: Halving n each iteration
--------------------------------------------------
Code Pattern:
for i in range(n):
    process(arr[i])

Complexity: O(n)
Reason: Single loop through n elements
--------------------------------------------------
Code Pattern:
for i in range(n):
    for j in range(n):
        process(arr[i], arr[j])
```

```
Complexity: O(n²)
Reason: Nested loops, n × n iterations
--------------------------------------------------
```

Big-O Family: Ω and Θ

Big-O
Upper bound (≤)

$$T(n) = O(f(n))$$

"At most this fast"

Big-Omega (Ω)
Lower bound (≥)

$$T(n) = \Omega(f(n))$$

"At least this slow"

Big-Theta (Θ)
Tight bound (=)

$$T(n) = \Theta(f(n))$$

"Exactly this rate"

Example: MergeSort - $T(n) = O(n \log n)$ ✓ (upper bound) - $T(n) = \Omega(n \log n)$ ✓ (lower bound)
- $T(n) = \Theta(n \log n)$ ✓ (tight bound)

🎯 Quick Quiz 3: Big-O Challenge

Match the code to its complexity:

```python
# Code A
for i in range(n):
    for j in range(i+1, n):
        process(i, j)

# Code B
i = n
while i > 0:
    process(i)
    i = i // 3

# Code C
for i in range(n):
    binary_search(arr, target)
```

Options: $O(n^2)$, $O(n \log n)$, $O(\log n)$

Answers: - Code A: $O(n^2)$ - nested loops with ~$n^2/2$ iterations - Code B: $O(\log n)$ - dividing by 3 each time - Code C: $O(n \log n)$ - n iterations × log n per search

Common Complexity Classes

Complexity	Name	Example Algorithm	1000 items
$O(1)$	Constant	Array access	1 op
$O(\log n)$	Logarithmic	Binary search	~10 ops
$O(n)$	Linear	Linear search	1,000 ops
$O(n \log n)$	Linearithmic	MergeSort	~10,000 ops
$O(n^2)$	Quadratic	Selection sort	1,000,000 ops
$O(2^n)$	Exponential	Subset generation	2^{1000} ops 😱

Key Insight: For large n, even small improvements in complexity class yield massive speedups!

Practice Problem: Analyzing Nested Loops

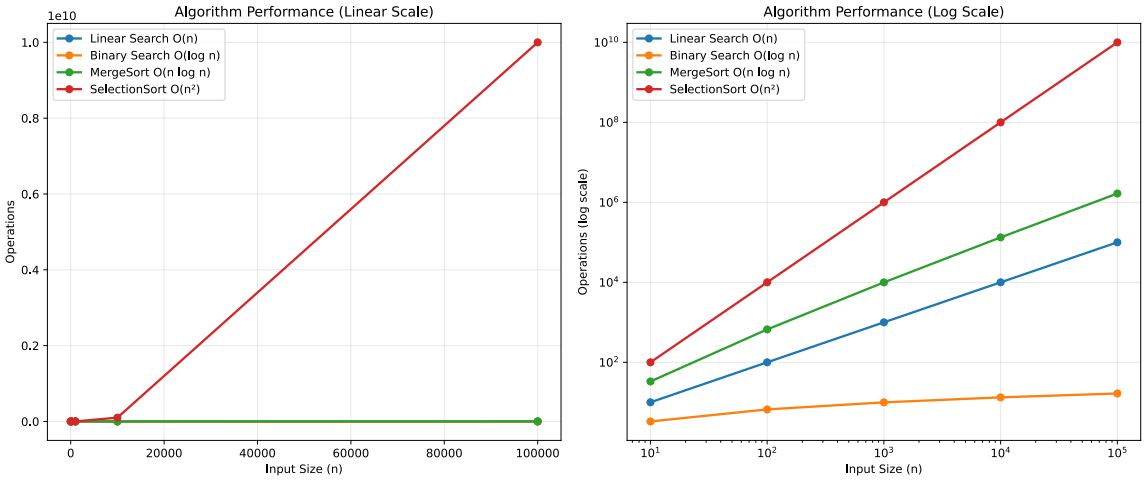
Analyze this code:

```
def mystery(n):
    count = 0
    for i in range(n):
        j = 1
        while j < n:
            count += 1
            j = j * 2
    return count
```

What's the complexity?

Solution: - Outer loop: n iterations - Inner loop: $\log_2(n)$ iterations (doubling j) - Total: $O(n \log n)$

Real-World Performance Impact



Lab Exercise: Implement & Analyze

Your Mission

1. **Implement** these sorting algorithms:
   - SelectionSort (O(n²))
   - MergeSort (O(n log n))
2. **Measure** actual running times for n = [100, 500, 1000, 5000]
3. **Plot** results and verify theoretical predictions

```
import time

def time_algorithm(algo, arr):
    start = time.time()
    algo(arr.copy())
    return time.time() - start

# Starter code
def selection_sort(arr):
    # TODO: Implement
    pass

def merge_sort(arr):
    # TODO: Implement
    pass
```

Key Takeaways 🎓

1. **Algorithms matter** - Can be difference between seconds and years!
2. **Divide & Conquer** - Break problems into smaller subproblems

3. **Asymptotic analysis** - Focus on growth rate, not exact counts
4. **Big-O notation** - Universal language for algorithm efficiency
5. **MergeSort beats $O(n^2)$** - $O(n \log n)$ is dramatically faster for large n

Next Week Preview

- QuickSort and randomized algorithms
- The Master Method for analyzing recursions
- Lower bounds and optimality

Discussion & Questions

Let's Discuss! 💬

Conceptual Questions

```
- Why does log n grow so slowly?
- When do constants matter?
- Is O(n) always better than O(n²)?
```

Practical Questions - Real-world algorithm choice? - Space vs. time tradeoffs? - Best practices for analysis?

Resources

- Visualizing Algorithms
- Big-O Cheat Sheet
- Practice problems on LeetCode

Homework & Practice

This Week's Assignments

1. **Reading**: Roughgarden Ch. 1-2 (complete)
2. **Implementation**:
   - Karatsuba multiplication
   - MergeSort with step counter
3. **Analysis Problems**:
   - Prove $T(n) = 3n^2 + 2n = O(n^2)$
   - Find tight bounds for 5 given functions
4. **Challenge**: Implement 3-way MergeSort

Next next class: QuickSort! 🚀

Remember: **Can we do better?**